On the new integro-differential nonlinear Volterra-Chandrasekhar equation

Ammar Khellaf[1,3] and Mohamed-Zine Aissaoui[2,3]

[1]Preparatory Class Department,National Polytechnic College of Constantine (Engineering College), Algeria
[2]Department of Mathematics, Faculty of Mathematics and Computer Science and Material Sciences, University 8 May 1945 of Guelma, Algeria
[3] Laboratoire de Mathematiques Appliquees et Modelisation (LMAM), University 8 May 1945
E-mail: amarlasix@gmail.com[1], aissaouizine@gmail.com[2]

Abstract

In this paper, we investigate the solution's existence and uniqueness for a new integro-differential nonlinear Volterra-Chandrasekhar equation. We approximate the solution of this equation by using Nyströme method. The accuracy and efficiency of this method are illustrated in some numerical examples.

2010 Mathematics Subject Classification. **45D05**. 65D07, 65R20.
Keywords. Nonlinear Voletrra integral equation, integro-differential equation, Chandrasekhar integral equation, fixed point, Nyströme method.

1 Introduction

Many physical processes and mathematical models usually lead to integro-differential nonlinear Volterra equations [1, 2]. These equations have attracted the attention of many researchers from the past to the present. In this paper, we study a new integro-differential nonlinear Volterra-Chandrasekhar equation, defined as follows:

$$u(t) = f(t) + k_1(t, u(t)) \int_0^t p(t-s)k_2(s, u(s), u'(s))ds, \quad t \in [0,1], \tag{1.1}$$

where f, p, k_1 and k_2 are given functions, u is the unknown. This equation type represents a convolution kernel [3].

Recall that the integro-differential nonlinear Volterra equation given by:

$$u(t) = f(t) + \int_0^t p(t-s)k_2(s, u(s), u'(s))ds, \quad t \in [0,1], \tag{1.2}$$

is recently studied in [4, 5, 6]. On the other hand, the nonlinear integral equation given by:

$$u(t) = f(t) + k_1(t, u(t)) \int_0^1 k_2(t, s, u(s))ds, \quad t \in [0,1], \tag{1.3}$$

is recently investigated analytically and numerically in [7, 8]. The last equation (1.3) is known in the literature as the Chandrasekhar quadratic integral equation [9, 10].

In this work, we investigate the equation (1.1) in analytical sense and in numerical sense. We prove that the existence of the solution by using ideas based on the Schauder's fixed point theorem,

Advanced Studies: Euro-Tbilisi Mathematical Journal Special Issue (10 - 2022), pp. 203–214.

Tbilisi Centre for Mathematical Sciences.

Received by the editors: 04 May 2021.
Accepted for publication: 27 October 2021.

as well as we show the uniqueness by utilizing similar techniques to those used in [5, 6], and finally we given an extension new theorem for the solution based on Krasnoselskii's fixed point theorem. Thereafter, In numerical sense we use the Nyström method [11, 12] to approximate the exact solution. Finally, we present a numerical application to show the accuracy of this method.

2 The framework

Let p, k_1 and k_2 be a real functions such that:

$$\begin{aligned}
&p : [-1,1] \to \mathbb{R},\ t \mapsto p(t),\\
&k_1 : [0,1] \times \mathbb{R} \to \mathbb{R},\ (t,x) \mapsto k_1(t,x),\\
&k_2 : [0,1] \times \mathbb{R}^2 \to \mathbb{R},\ (s,x,y) \mapsto k_2(s,x,y).
\end{aligned}$$

We assume that p, k_1 and k_2 satisfy the following conditions:

$$(H1)\left\|\begin{array}{l}
\text{i) } \dfrac{\partial k_1}{\partial t}, \dfrac{\partial k_1}{\partial x} \in \mathcal{C}([0,1]\times\mathbb{R}),\\
\forall s \in [0,1], \quad k_2(s,\cdot,\cdot) \in \mathcal{C}(\mathbb{R}^2)\\
\text{ii) } p \in \mathcal{C}^1([-1,1]), \quad p(0) = 0,\\
\text{iii) } \exists M > 0,\ \forall t,s \in [0,1],\ \forall x,y \in \mathbb{R} :\\
\max\left\{|k_1(t,x)|, |\dfrac{\partial k_1}{\partial t}(t,x)|, |\dfrac{\partial k_1}{\partial x}(t,x)|, |p(t-s)k_2(s,x,y)|, |p'(t-s)k_2(s,x,y)|\right\} \leq M.
\end{array}\right.$$

We define the superposition operator K_1 generated by the function k_1 (see [13]), given by the formula

$$K_1(u)\,(t) = k_1(t,u(t)), \quad t \in [0,1].$$

In what follows, we do not use the monotonicity properties of the superposition operator K_1 to investigate the solvability of the equation (1.1). However, in [14] the monotonicity properties has been applied to investigate the solvability of the equation (1.3).

For $f \in \mathcal{C}^1(0,1)$, we consider again the integro-differential nonlinear Volterra-Chandrasekhar

$$\forall t \in [0,1],\ u(t) = k_1(t,u(t)) \int_0^t p(t-s)k_2(s,u(s),u'(s))ds + f(t), \tag{2.1}$$

The natural framework of our solution u is the separable Banach space $\mathcal{C}^1([0,1])$. If we derive the two sides of this equation, we obtain for $t \in [0,1]$:

$$\begin{aligned}
u'(t) = \quad & \left[\frac{\partial k_1}{\partial t}(t,u(t)) + u'(t)\frac{\partial k_1}{\partial x}(t,u(t))\right] \int_0^t p(t-s)k_2(s,u(s),u'(s))ds+ \\
& k_1(t,u(t)) \int_0^t p'(t-s)k_2(s,u(s),u(s))ds + f'(t).
\end{aligned} \tag{2.2}$$

Our goal is to find conditions that ensure the existence and uniqueness of the solution, then build a numerical technique to approach this solution.

3 Analytical study

Let $T_1 = \dfrac{1}{1+M^2}$, for $f \in \mathcal{C}^1([0,1])$ we define the functional Φ_f such that

$$\forall \xi \in \mathcal{C}^1([0,T_1]),\, \forall t \in [0,T_1],\, \Phi_f(\xi)(t) = k_1(t,\xi(t)) \int_a^t p(t-s) k_2(s,\xi(s),\xi'(s))ds + f(t).$$

We define also the set

$$\mathcal{F} = \left\{ \xi \in \mathcal{C}^1([0,T_1]) : \xi(0) = f(0),\, \xi'(0) = f'(0),\, \max_{t\in[0,T_1]} |\xi(t)| \leq c_1,\, \max_{t\in[0,T_1]} |\xi'(t)| \leq c_2 \right\},$$

where

$$c_1 = \max_{t\in[0,1]} |f(t)| + M^2, \quad c_2 = \max_{t\in[0,1]} |f'(t)| + M^2 \left(2 + \max_{t\in[0,1]} |f'(t)|\right).$$

Proposition 3.1. For $f \in \mathcal{C}^1([0,1])$, Φ_f is continuous from $\mathcal{F}$ to $\mathcal{C}^1([0,T_1])$.

Proof. Let $\xi \in \mathcal{F}$, it is clear that $\Phi_f(\xi)(\cdot)$ is continuous on $[0,T_1]$. In addition, for all $t \in [0,T_1]$:

$$\begin{aligned} \Phi_f(\xi)'(t) &= \left[\frac{\partial k_1}{\partial t}(t,\xi(t)) + \xi'(t)\frac{\partial k_1}{\partial x}(t,\xi(t))\right] \int_0^t p(t-s)k_2(s,\xi(s),\xi'(s))ds + \\ &\quad k_1(t,\xi(t)) \int_0^t p'(t-s)k_2(s,\xi(s),\xi'(s))ds + f'(t), \end{aligned}$$

which is continuous on $[0,T_1]$ and bounded.
Let $(\xi_n)_{n\in\mathbb{N}}$ a sequence in $\mathcal{F}$ converging to ξ. Then, for all $t \in [0,T_1]$:

$$\begin{aligned} \lim_{n\to+\infty} \Phi_f(\xi_n)(t) &= \lim_{n\to+\infty} \left(k_1(t,\xi_n(t)) \int_a^t p(t-s)k_2(s,\xi_n(s),\xi_n'(s))ds + f(t) \right) \\ &= \left(k_1(t, \lim_{n\to+\infty} \xi_n(t)) \int_a^t p(t-s)k_2(s, \lim_{n\to+\infty} \xi_n(s), \lim_{n\to+\infty} \xi_n'(s))ds + f(t) \right) \\ &= \Phi_f(\xi)(t). \end{aligned}$$

Q.E.D.

3.1 Existence

For the existence of the solution, we add the following condition:

$$(H2) \left\| \begin{array}{l} i)\, \exists A > 0,\, \forall t \in [0,1],\, \forall x, \overline{x} \in \mathbb{R}, \quad |k_1(t,x) - k_1(t,\overline{x})| \leq A|x - \overline{x}|, \\ ii)\, \exists C > 0, \forall u \in \mathcal{C}^1([0,1]),\, \forall t,t' \in [0,1], \quad |\dfrac{dK_1}{dt}(t,u(t)) - \dfrac{dK_1}{dt}(t',u(t'))| \leq C|t-t'|. \end{array} \right.$$

This condition is similar to the assumptions stated in [7, 8].

Theorem 3.2. Eq. (2.1) defined on $[0,T_1]$ has a solution in $\mathcal{C}^1([0,T_1])$.

Proof. It is easy to check that the set $\mathcal{F}$ is closed and convex.
For all $\xi \in F$ we can prove that

$$\Phi_f(\xi)(0) = f(0),\ \Phi'_f(\xi)(0) = f'(0),\ \max_{t\in[0,T_1]} |\Phi_f(\xi)(t)| \leq c_1,$$

and

$$\begin{aligned}\max_{t\in[0,T_1]} |\Phi_f(\xi)'(t)| &\leq 2T_1M^2 + c_2M^2T_1 + \max_{t\in[0,1]} |f'(t)| \\ &\leq \frac{1}{1+M^2}\left[2M^2 + M^2c_2 + (1+M^2)\max_{t\in[0,1]} |f'(t)|\right] = c_2.\end{aligned}$$

Then, $\Phi_f(\mathcal{F}) \subset \mathcal{F}$. Furthermore, Φ_f is continuos on $\mathcal{F}$ (see proposition 3.1). Consecutively, for $\xi \in F$ and for all $t, t' \in [0, T_1]$:

$$\begin{aligned}&|\Phi_f(\xi)(t) - \Phi_f(\xi)(t')| \leq |k_1(t', \xi(t'))| \int_t^{t'} |p(t-s)k_2(s, \xi(s), \xi'(s))|\, ds + \\ &+ \int_0^t |p(t-s)k_2(s, \xi(s), \xi'(s))|\, ds\ |k_1(t, \xi(t)) - k_1(t', \xi(t'))| + \max_{t\in[0,1]} |f'(t)|\ |t-t'| \\ &\leq (M^2 + \max_{t\in[0,1]} |f'(t)|)|t-t'| + M\,|k_1(t, \xi(t)) - k_1(t, \xi(t'))| + M\,|k_1(t, \xi(t')) - k_1(t', \xi(t'))| \\ &\leq (M^2 + \max_{t\in[0,1]} |f'(t)|)|t-t'| + AM\,|\xi(t) - \xi(t')| + M \max_{s\in[0,1]} \left\{\left|\frac{\partial k_1}{\partial t}(s, \xi(s))\right|\right\} |t-t'| \\ &\leq (2M^2 + \max_{t\in[0,1]} |f'(t)|)|t-t'| + AM \max_{t\in[0,T_1]} |\xi'(t)|\ |t-t'| \\ &\leq (2M^2 + AMc_2 + \max_{t\in[0,1]} |f'(t)|)|t-t'|.\end{aligned}$$

Further, we can find similarly that

$$|\Phi_f(\xi)(t) - \Phi_f(\xi)(t')| \leq C_1|t-t'|,$$

where C_1 is a positive constant. Thus, using Schauder's theorem, we obtain that Φ_f has a fixed point in $\mathcal{F}$. Q.E.D.

3.2 Uniqueness

We add the following conditions:

$$(H3)\left\|\begin{array}{ll} & \exists B, \overline{B}, D, \overline{D} \in \mathbb{R}_+,\ \forall x, \overline{x}, y, \overline{y} \in \mathbb{R},\ \forall t, s \in [0,1] \\ i) & |k_2(s,x,y) - k_2(s,\overline{x},\overline{y})| \leq B|x-\overline{x}| + \overline{B}|y-\overline{y}|, \\ ii) & |\dfrac{\partial k_1}{\partial x}(t,x) - \dfrac{\partial k_1}{\partial x}(t,\overline{x})| \leq D|x-\overline{x}|, \\ iii) & |\dfrac{\partial k_1}{\partial t}(t,x) - \dfrac{\partial k_1}{\partial t}(t,\overline{x})| \leq \overline{D}|x-\overline{x}|. \end{array}\right.$$

These conditions are also similar to assumptions stated in [4, 5, 6, 7, 8] to obtain the uniqueness of the equations (1.2) and (1.3).

To show the uniqueness, we need an intermediate result.

Lemma 3.3. If φ is a continuous positive function on $[a,b]$, which verifies

$$\exists p>0,\, \forall t\in[a,b],\, \varphi(t)\le p\int_a^t \varphi(s)ds.$$

Then, $\varphi=0$.

Proof. There exists $\mu>0$ such that

$$\forall t\in[a,b],\, \varphi(t)\le\mu.$$

Then,

$$\varphi(t)\le p\mu\int_a^t ds = p\mu(t-a).$$

On the other hand, we have

$$\varphi(t)\le p\int_a^t \varphi(s)ds.$$

Therefore,

$$\varphi(t)\le p^2\mu\int_a^t (s-a)^2 ds = p^2\mu\frac{(t-a)^2}{2}.$$

Repeat n times this operation, we find

$$\varphi(t)\le p^n\mu\frac{(t-a)^n}{n!}\longrightarrow 0,\, n\to\infty.$$

Q.E.D.

Consider $\tilde{T}_1=\min\left\{T_1,\dfrac{1}{AM+1}\right\}$, the next theorem gives us the uniqueness of the solution of the equation (2.1) on $[0,\tilde{T}_1]$.

Theorem 3.4. The solution of the Eq. (2.1) is unique on $[0,\tilde{T}_1]$.

Proof. Suppose that $u,v\in\mathcal{F}$ are two solutions of the equation (2.1) on $[0,\tilde{T}_1]$.
Set

$$\gamma(t)=|u(t)-v(t)|+|u'(t)-v'(t)|,\quad t\in[0,\tilde{T}_1].$$

Using the conditions $(H2)$ and $(H3)$, we can prove that

$$(1-AM\tilde{T}_1)|u(t)-v(t)|\le C_1\int_0^t\gamma(s)ds,$$

$$(1-M^2\tilde{T}_1)|u'(t)-v'(t)|\le C_2|u(t)-v(t)|+C_3\int_0^t\gamma(s)ds,$$

where C_1, C_2 and C_2 are positive constants depending only on $A,B,\overline{B},D,\overline{D}$ and M. Therefore, there exists $p>0$ such that

$$\forall t\in[0,\tilde{T}_1],\quad \gamma(t)\le p\int_0^t\gamma(s)ds.$$

Using the previous lemma, we obtain: $u=v$. Q.E.D.

3.3 Extension of the solution

In the previous subsections, we showed that the equation (2.1) on $[0, \tilde{T}_1]$ has a unique solution. in what follows, we prove that this solution can be extended on $[0, 1]$.

Let u_1 denoted the solution of the equation (2.1) on $[0, \tilde{T}_1]$. Let $\tilde{T}_2 > \tilde{T}_1$ such that $\tilde{T}_2 - \tilde{T}_1 = d$. For all $t \in [\tilde{T}_1, \tilde{T}_2]$, we set the function u_2 satisfies the following equation:

$$\begin{aligned} u_2(t) &= k_1(t, u_2(t)) \int_{\tilde{T}_1}^{t} p(t-s) k_2(s, u_2(s), u_2'(s)) ds + f(t) \\ &\quad + k_1(t, u_2(t)) \int_0^{\tilde{T}_1} p(t-s) k_2(s, u_1(s), u_1'(s)) ds, \end{aligned} \tag{3.1}$$

If u_2 is the unique solution of the equation (3.1), then the equation (2.1) has a unique solution u on $[0, \tilde{T}_2]$ given by:

$$u(t) = \begin{cases} u_1(t) & 0 \le t \le \tilde{T}_1, \\ u_2(t) & \tilde{T}_1 < t \le \tilde{T}_2, \end{cases}$$

However, to prove that the equation (3.1) has a unique solution, we assume in addition to $(H2)$ that:

$$(H'2) \left\| \begin{array}{l} i)\ \exists A > 0,\ \forall t \in [0,1],\ \forall x, \overline{x} \in \mathbb{R}:\ |k_1(t,x) - k_1(t,\overline{x})| \le A|x - \overline{x}|. \\ \\ ii)\ A < 1. \end{array} \right.$$

Theorem 3.5. Eq. (3.1) has solution on $[\tilde{T}_1, \tilde{T}_2]$.

Proof. Let $u \in \mathcal{C}^1([\tilde{T}_1, \tilde{T}_2])$, we define the operators S and T by the relations:

$$S(u)(t) = k_1(t, u(t)) \int_{\tilde{T}_1}^{t} p(t-s) k_2(s, u(s), u'(s)) ds + f(t), \tag{3.2}$$

$$T(u)(t) = k_1(t, u(t)) \int_0^{\tilde{T}_1} p(t-s) k_2(s, u_1(s), u_1'(s)) ds. \tag{3.3}$$

If $d = \tilde{T}_1$, then by using the same technics as in the previous sections, the operator S has fixed point based on Schauder's fixed point theorem. On the other Side, the operator T is a contraction on $\mathcal{C}^1([\tilde{T}_1, \tilde{T}_2])$. For all $u, v \in \mathcal{C}^1([\tilde{T}_1, \tilde{T}_2])$, for $t \in [\tilde{T}_1, \tilde{T}_2]$:

$$\begin{aligned} |T(u)(t) - T(v)(t)| &\le \int_0^{\tilde{T}_1} |p(t-s) k_2(s, u_1(s), u_1'(s))| ds |k_1(t, u(t)) - k_1(t, v(t))| \\ &\le \tilde{T}_1 M^2 A |u(t) - v(t)| \\ &\le A |u(t) - v(t)|. \end{aligned}$$

Furthermore, for all $u, v \in \mathcal{C}^1([\tilde{T}_1, \tilde{T}_2])$, we have $S(u) + T(v) \in \mathcal{M}$ where the set $\mathcal{M}$ is:

$$\mathcal{M} = \{u \in \mathcal{C}^1([\tilde{T}_1, \tilde{T}_2]) : |u(t)| \le 2\tilde{T}_1 M^3 + \max |f|\}.$$

So, according to Krasnoselskii's fixed point theorem the equation $u = S(u) + T(u)$ has a fixed point u_2 in $\mathcal{C}^1([\tilde{T}_1, \tilde{T}_2])$. Q.E.D.

4 Numerical study

Under the conditions $(H1), (H2), (H'2)$ and $(H3)$, we have shown that the Eq. (2.1) has a unique solution in $\mathcal{C}^1([0,1])$. In this section, we will present a numerical method based on the numerical integration to approximate the solution. This numerical integration method is called Nyströme method (see [11, 12]).

Let $N \in \mathbb{N}$, on the interval $[0,1]$, we define the subdivision:

$$t_j = jh,\ h = \frac{1}{N},\ 0 \leq j \leq N.$$

The numerical integration formula is given by:

$$\int_0^1 \xi(t) \backsimeq h \sum_{i=0}^{N} w_i \xi(t_i), \tag{4.1}$$

where, w_i are real number verifying

$$W = \sup_{N\in\mathbb{N}} \sum_{i=0}^{N} |w_i| < \infty.$$

Using this quadrature method on the Eqs. (2.1) and (2.2), we obtain the following system

$$\begin{aligned}
U_0 = &\quad f(0),\\
V_0 = &\quad f'(0),\\
U_n = &\quad f(t_n) + h\, k_1(t_n, U_n) \sum_{i=0}^{n-1} w_i p(t_n - t_i) k_2(t_i, U_i, V_i),\\
V_n = &\quad f'(t_n) + [\frac{\partial k_1}{\partial t}(t_n, U_n) + V_n \frac{\partial k_1}{\partial x}(t_n, U_n)] h \sum_{i=0}^{n-1} w_i p(t_n - t_i) k_2(t_i, U_i, V_i)\\
&\quad + h k_1(t_n, U_n) \sum_{i=0}^{n} w_i p'(t_n - t_i) k_2(t_i, U_i, V_i).
\end{aligned} \tag{4.2}$$

where, U_n approaches $u(t_n)$ and V_n approaches $u'(t_n)$ for $0 \leq n \leq N$.

4.1 System study

Theorem 4.1. For h sufficiently small, the system (4.2) has a unique solution.

Proof. Suppose that the space $\mathbb{R}^2$ has the following norm:

$$\forall \begin{pmatrix} X \\ Y \end{pmatrix} \in \mathbb{R}^2, \quad \left\| \begin{pmatrix} X \\ Y \end{pmatrix} \right\|_1 = |X| + |Y|.$$

For all $n \geq 1$, we set:

$$\Psi_n \begin{pmatrix} X \\ Y \end{pmatrix} := \begin{pmatrix} f(t_n) + h\alpha_{n-1} k_1(t_n, X), \\ f'(t_n) + h\alpha_{n-1}[\frac{\partial k_1}{\partial t}(t_n, X) + Y \frac{\partial k_1}{\partial x}(t_n, X)] + hp'(0) k_1(t_n, X) k_2(t_n, X, Y) + h\beta_{n-1} k_1(t_n, X). \end{pmatrix}$$

where,

$$\alpha_{n-1} = \sum_{i=0}^{n-1} w_i p(t_n - t_i) k_2(t_i, U_i, V_i),$$
$$\beta_{n-1} = \sum_{i=0}^{n-1} w_i p'(t_n - t_i) k_2(t_i, U_i, V_i).$$

Under the conditions $(H1) - (H3)$, we get:

$$\forall \begin{pmatrix} X \\ Y \end{pmatrix}, \begin{pmatrix} X' \\ Y' \end{pmatrix} \in \mathbb{R}^2, \left\| \Psi_n \begin{pmatrix} X \\ Y \end{pmatrix} - \Psi_n \begin{pmatrix} X' \\ Y' \end{pmatrix} \right\| \leq hC \left\| \begin{pmatrix} X \\ Y \end{pmatrix} - \begin{pmatrix} X' \\ Y' \end{pmatrix} \right\|_2$$

where, C is a positive constant. Then, for h sufficiently small, by using Banach's theorem, we obtain the result. Q.E.D.

4.2 Error analysis

In this section, we will show that the approximate solutions constructed in the previous section converge to the exact solutions of the Eqs (2.1) and (2.2). For this, we define for all $n \in \mathbb{N}$

$$\varepsilon_n := |U_n - u(t_n)| + |V_n - u'(t_n)|.$$

We say that the method presented by the system (4.2) is convergent, if

$$\lim_{h \to 0} \left(\max_{0 \leq n \leq N} \varepsilon_n \right) = 0.$$

Let $\xi \in \mathcal{C}^1([0,1])$, for $n \in \mathbb{N}$, we define the local consistency error as:

$$\begin{aligned} \delta(h, t_n) \quad = \quad & \left| \int_0^{t_n} p(t_n - s) K_2(s, \xi(s), \xi'(s)) ds - h \sum_{i=0}^{n-1} w_i p(t_n - t_i) K_2(t_i, \xi(t_i), \xi'(t_i)) \right| \\ & + \left| \int_0^{t_n} p'(t_n - s) K_2(s, \xi(s), \xi'(s)) ds - h \sum_{i=0}^{n} w_i p'(t_n - t_i) K_2(t_i, \xi(t_i), \xi'(t_i)) \right|. \end{aligned}$$

We say that the approximation method (4.2) is consistent with the equations (2.1) and (2.2), if

$$\forall \xi \in \mathcal{C}^1([0,1]), \quad \lim_{h \to 0} \left(\max_{0 \leq n \leq N} \delta(h, t_n) \right) = 0.$$

The next lemma is intermediate result using to prove that the method presented in (4.2) is convergente.

Lemma 4.2. Let $a_0, a_1, \ldots$ satisfy

$$|a_n| \leq A \sum_{i=0}^{n-1} |a_i| + B,$$

where $A > 0$, $B > 0$. Then

$$|a_n| \leq (1 + A)^{n-1} (B + A|a_0|).$$

Proof. The result is easily establish with an inductive argument. Q.E.D.

Theorem 4.3. If the approximation method given by (4.2) is said to be consistent with (2.1) and (2.2). Then

$$\lim_{h\to 0}\left(\max_{0\le n\le N}\varepsilon_n\right)=0.$$

Proof. For $n\ge 1$, we get

$$|U_n-u(t_n)|\le M\delta(h,t_n)+hMAW|U_n-u(t_n)|+hM^2W\max\{B,\overline{B}\}\sum_{j=0}^{n-1}\varepsilon_j.$$

Then, we evaluate the difference $|V_n-u'(t_n)|$, to find for all h small enough,

$$\varepsilon_n \quad\le\quad a(h)+b(h)\sum_{j=0}^{n-1}\varepsilon_j,$$

where

$$\begin{cases} a(h)=M(3+L)\delta(h,t_n),\\ \\ b(h)=h(3+L)M^2W\max\{B,\overline{B}\},\\ \\ \alpha=1-h\max\{MW(2A+D+\overline{D}+1),4M^2W\}.\end{cases}$$

Using the lemma 4.2, we get

$$\begin{aligned}\varepsilon_n \quad\le\quad & \frac{1}{\alpha}\left(1+\frac{h(3+L)M^2W\max\{B,\overline{B}\}}{\alpha}\right)^{n-1}\times\\ & \left(M(3+L)\max_{0\le i\le n}\delta(h,t_i)+h(3+L)M^2W\max\{B,\overline{B}\}\varepsilon_0\right).\end{aligned}$$

But,

$$\left(1+\frac{h(3+L)M^2W\max\{B,\overline{B}\}}{\alpha}\right)^{n-1}\le\left(1+\frac{(3+L)M^2W\max\{B,\overline{B}\}}{N\alpha}\right)^{N},$$

and

$$\lim_{N\to+\infty}\left(1+\frac{(3+L)M^2W\max\{B,\overline{B}\}}{N\alpha}\right)^{N}<\infty.$$

Thus, there exists $\theta>0$ such that:

$$\forall N\in\mathbb{N},\ \max_{1\le n\le N}\frac{1}{\alpha}\left(1+\frac{h(3+L)M^2W\max\{B,\overline{B}\}}{\alpha}\right)^{n-1}\le\theta.$$

And the desired result is obtained. Q.E.D.

Table 1. The numerical results for Eq. (5.1)

N	$\max\limits_{0\le i\le N} \lvert U_i - u(t_i)\rvert$	$\max\limits_{0\le i\le N} \lvert V_i - u'(t_i)\rvert$
10	4.0000E-03	3.7000E-03
50	7.9528E-04	7.3317E-04
100	3.9793E-04	3.6633E-04
200	1.9904E-04	1.8317E-04
500	7.9635E-05	7.3253E-05
1000	3.9821E-05	3.6624E-05
1500	2.6548E-05	2.4416E-05

5 Numerical result

We use the trapezoidal method since it guarantees that the approximation method given by (4.2) is consistent with the Eqs (2.1) and (2.2).

The terms U_n and V_n will not be exactly calculated. They will be approached using Banach's iteration method with a stopping criteria of type

$$\|X_{new} - X_{old}\| \le \frac{1}{10N}.$$

Example 1: We choose the following equations

$$u(t) = \frac{t}{1+u^2(t)} \int_0^t \frac{(t-s)}{5+(u(s)+u'(s))^2} ds + f(t), \quad t \in [0,1]. \tag{5.1}$$

The functions $K_1(t,x) = \dfrac{t}{1+x^2}$, $K_2(s,x,y) = \frac{1}{5+(x+y)^2}$ and $p(s) = s$ satisfy the assumptions $(H1)$, $(H2)$ and $(H3)$ with the constants:

$$M = 2, \quad A, B, \overline{B}, D, \overline{D} \le 3.$$

If we take:

$$f(t) = t + \left[t \left(\frac{log(t^2+2t+6)}{2} - \frac{log(6)}{2} + \frac{1}{5}(5^{\frac{1}{2}}(arctan(5^{\frac{-1}{2}}) - arctan(\frac{(5^{\frac{1}{2}}t)}{5} + 5^{\frac{-1}{2}})) \right) + \left(5^{\frac{1}{2}} t (arctan(5^{\frac{-1}{2}}) - arctan((5^{\frac{1}{2}}t)\frac{1}{5} + 5^{\frac{-1}{2}})))\frac{1}{5} \right) \right] \frac{1}{t^2+1},$$

Then, we get $u(t) = t$.

6 Conclusion

We have built assumptions that guarantee solution's existence and uniqueness for a generalized integro-differential nonlinear Volterra equation. Developed numerical example shows the effectiveness of the Nyström method used to approximate the solution of this equation. As perspective, we will study the case where the kernel is weakly singular using different method as in [15, 16].

Acknowledgment

All the authors thank the General Direction of Scientific Research and Technological Development (DGRSDT) of the Ministry of Higher Education and Scientific Research of Algeria.

References

[1] D. O'Regan and M. Meehan, *Existence theory for nonlinear integral and integro-differential equations*, Kluwer academic, Dordrecht (1998).

[2] T.A. Burton, *Volterra integral and differential equation*, academic press, New york (1983).

[3] H. Brunner and PJ. van der Houwen, *The numerical solution of Volterra equations CWI monographs*, North-Holland, Amsterdam (1986).

[4] M. Ghiat and H. Guebbai, *Analytical and numerical study for an integro-differential nonlinear volterra equation with weakly singular kernel*, Comp. Appl. Math**37** (2018) 4661–4674.

[5] S. Segni, M. Ghiat and H. Guebbai, *New approximation method for Volterra nonlinear integro-differential equation*, Asian-European Journal of Mathematics **12** (2019) 1950016.

[6] H. Guebbai, M.Z Aissaoui, I. Debbar and B. Khalla, *Analytical and numerical study for an integro-differential nonlinear Volterra equation*, Appl. Math. Comp**229** (2014) 367-373.

[7] K. Maleknejad, P. Torabi and S. Sauter, *Numerical solution of a non linear Volterra integral equation*, Vitnam J. Math **44** (2016) 5–28.

[8] K. Maleknejad, P. Torabi and R. Mollapourasl, *Fixed point method for solving nonlinear quadratic Volterra integral equations*, Computer and Mathematics with Applications **44** (2011) 5–28.

[9] S. Chandrasekhar, *Radiative Transfar*, Dover, New york (1960).

[10] J. Caballero, A.B. Mingarelli and K. Sadarangani, *Existence of solutions of an integral equation of Chandrasekhar type in the theory of radiative, Electron*, J. Diff. Equat **57** (2006) 1–11.

[11] K. Atkinson and W. Han, *Theoretical Numerical Analysis: A Functional Analysis Framework*, Springer-Verlag, New York (2001).

[12] P. Linz, *Analytical and Numerical Methods for Volterra Equations. SIAM Studies in Applied Mathematics* Philadelphia (1985).

[13] J. Appell and P.P.Zabrejko, *Nonlinear superposition operator*, Combridge Tracts in Math. Cambridge Univ. Press (1990).

[14] J. Banas and K. Sadarangani, *Monotonicity properties of the superposition operator and thier applications*, J. Math. Anal. Appl **340** (2008) 1385–1394.

[15] A. Khellaf, W. Merchela and S. Benarab, New numerical process solving nonlinear infinite dimensional equations. Computational and Applied Mathematics, **1** (2020) 1–15.

[16] A.Khellaf, M.Z Aissaoui, New theoretical conditions for solving functional nonlinear equations by linearization then discretization. Int. J. Nonlinear Anal. Appl. **13** (2022) 1, 2857-2869